JOURNAL OF FUNCTIONAL ANALYSIS **151**, 50–76 (1997)
ARTICLE NO. FU973120

A Classification of Simple Limits of Splitting Interval Algebras

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Received February 29, 1996; accepted March 28, 1996; revised March 28, 1996

Let A be a unital simple limit of finite direct sums of sub-homogeneous interval algebras of a certain type (cf. Definition 1.1). It is proved that A can be classified by the scaled ordered group $K_0(A)$, the simplex $T(A)$, and the canonical pairing between them. It is also shown that $K_0(A)$ might fail to have the Riesz decomposition property. © 1997 Academic Press

0. INTRODUCTION

This paper is a contribution to the recent C^*-algebra classification program initiated by George A. Elliott. Until now the K_0 groups of classified C^*-algebras had all enjoyed the so-called Riesz decomposition property (see [E3] for a survey). In this paper we classify a class of simple C^*-algebras whose K_0-groups might fail to have this property.

MAIN THEOREM. *Let A, B be two simple unital inductive limits of finite direct sums of splitting interval algebras. If there is a homomorphism*

$$\kappa: (K_0(A), K_0(A)^+, [1]) \to (K_0(B), K_0(B)^+, [1])$$

of scaled ordered groups and a continuous affine map $\theta: T(B) \to T(A)$ of the tracial state spaces that are compatible with respect to the pairing between K_0-groups and traces, then there exists a unital $*$-homomorphism $\rho: A \to B$ which induces κ and θ.

Moreover, if κ and θ are isomorphisms, then ρ can be chosen to be an isomorphism.

Besides providing new range for the invariants, the inductive limits of subhomogenous algebras are interesting for many other reasons (See [Su] for a good indication). This paper is a continuation of [EGJS] which classified simple C^*-algebras that can be expressed as inductive limits of

* We are grateful to Dr. G. A. Elliott for his support. The second author thanks Dr. G. Gong for a discussion on the lack of the Riesz decomposition property for the K_0 of some splitting interval algebras.

0022-1236/97 $25.00

finite direct sums of matrix algebras over dimension drop algebras (also see [T3]). A matrix algebra over a dimension drop algebra can be viewed as a matrix algebra over a subhomogenous interval. The building blocks used in this paper (matrix algebras over non-Hausdorff intervals) are another kind of subhomogenous algebras. It would be interesting to combine splitting interval algebras used here and the other building blocks used in other classifications to classify more C^*-algebras. We note that G. A. Elliott and K. Thomsen ([E4], [ET] and [T2]) have studied the ranges of the invariant for simple C^*-algebras constructed using more general building blocks than those used in this paper.

Our proof is an intertwining argument. However, special attention must be paid to the fact that the spectrum of each splitting interval algebra may not be Hausdorff, which in particular causes difficulty in establishing the so called existence theorem in Section 3. This difficulty will be circumvented through a structure theory of compatible pairs in Section 2 and Section 3. The analysis in Section 2 also leads to a result about almost compatible pairs that we shall need in the very beginning of our proof. In Section 4 we shall establish the local uniqueness result which will be needed in the last part of our proof. In the last section, Section 6, we construct a simple unital inductive limit of finite direct sum of splitting interval algebras, whose K_0 group does not have Riesz decomposition property. It was shown in [Zh] that the K_0-groups of real rank zero C^*-algebras had Riesz decomposition property. In fact, all the other C^*-algebras that have been classified so far have this property.

Unless specified otherwise, C^*-algebras and C^*-homomorphisms in this paper are unital. For two C^*-algebras A and B, a morphism (or a map, for short) $\phi_0\colon K_0(A) \to K_0(B)$ is a homomorphism of scaled ordered groups, and a morphism $\phi_T\colon T(B) \to T(A)$ is a continuous affine map.

1. PRELIMINARIES

The basic building blocks for the C^*-algebras to be studied in this paper are C^*-subalgebras of matrix algebras over the interval which split at the endpoints.

1.1. Definition. A splitting interval algebra is any C^*-algebra of the form

$$\mathscr{S}(\bar{n}_0\,;\bar{n}_1)=\left\{f\in M_n(C[0,1]):f(x)\in\bigoplus_{i=1}^{r_x} M_{n_{x_i}}(\mathbb{C}),\ x=0, \text{ or } 1\right\},$$

where each $\bar{n}_x=(n_{x_1}, ..., n_{x_{r_x}})$, for $x=0$ or 1, is a partition of n (by positive integers).

If $\bar{n}_0 = \bar{n}_1 = n$, then $\mathcal{S}(\bar{n}_0 ; \bar{n}_1) = M_n(C[0, 1])$.

In general, a splitting interval algebra $A = \mathcal{S}(\bar{n}_0 ; \bar{n}_1)$ is a continuous field of C^*-algebras over $[0, 1]$, whose fibre $A(x)$ at x is the full matrix algebra $M_n(\mathbb{C})$ unless $x = 0$ or 1, where the fibres are

$$A(0) = \bigoplus_{i=1}^{r_0} M_{n_{0_i}}(\mathbb{C}), \qquad \text{and} \qquad A(1) = \bigoplus_{i=1}^{r_1} M_{n_{1_i}}(\mathbb{C}),$$

respectively. $x = 0$ or 1 will be called a broken endpoint if $r_x > 1$, that is, if $A(x)$ is not $M_n(\mathbb{C})$.

It is easy to check that the spectrum $sp(A)$ of A is $\{0_1, ..., 0_{n_0}\} \cup (0, 1) \cup \{1_1, ..., 1_{n_1}\}$, with the natural non-Hausdorff topology. It has a canonical quotient map $[\cdot]: sp(A) \to [0, 1]$. If $x = 0$ or 1 is a broken endpoint, then any of $\{x_i : 1 \leqslant i \leqslant r_x\}$, will be called a fractional endpoint. It is sometimes useful to think of A as a field over $sp(A)$, where the fibre $A(x_i)$ over a fractional endpoint x_i is $M_{x_i}(\mathbb{C})$. Let $Q_{x_i}: A \to A(x_i)$ be the canonical evaluation map at an endpoint x_i. It induces a morphism:

$$(Q_{x_i})_* : K_0(A) \to K_0(A(x_i)) \cong \mathbb{Z}.$$

The following results are well-known and easy to check:

1.2. Lemma. (1) *The direct sum* $\bigoplus_{x_i} (Q_{x_i})_* : K_0(A) \to \mathbb{Z}^{r_0 + r_1}$ *is an injective morphism, and identifies* $K_0(A)$ *with the subgroup of* $\mathbb{Z}^{r_0+r_1}$:

$$\left\{ (\bar{k}_0 ; \bar{k}_1) \in \mathbb{Z}^{r_0} \times \mathbb{Z}^{r_1} : \sum_{i=1}^{r_0} k_{0_i} = \sum_{i=1}^{r_1} k_{1_i} \right\}$$

with the inherited order from the standard order of $\mathbb{Z}^{r_0+r_1}$, *where* k_{x_i}*'s are coordinates of* $\bar{k}_x$. *Also*, $[1] \cong (\bar{n}_0 ; \bar{n}_1)$.

(2) $K_1(A) = \{0\}$.

In particular, $K_0(\mathcal{S}(\bar{n}_0; \bar{n}_1)) \cong \mathbb{Z}^{r_0 + r_1 - 1}$ as a group.

1.3. Lemma. (1) *Any Radon probability measure* μ *on* $[0, 1]$ *defines a tracial state on* A *in the following way*:

$$\mu(f) = \int \mathrm{Tr}(f)\, d\mu, \qquad \text{for} \quad f \in A,$$

where Tr *is the normalized canonical trace on* $M_n(\mathbb{C})$, *the generic fibre of* A. *The corresponding tracial state will be denoted again by* μ.

(2) *Any* x_i *defines a point-mass tracial states on* A,

$$\delta_{x_i}(f) = \mathrm{Tr}_{A(x_i)}(Q_{x_i}(f)),$$

where $\mathrm{Tr}_{A(x_i)}$ *is the normalized trace on* $A(x_i)$.

(3) $\delta_0 = \sum_i (n_{0_i}/n)\, \delta_{0_i}$ *and* $\delta_1 = \sum_i (n_{1_i}/n)\, \delta_1$, *where* δ_0 *and* δ_1 *are defined in* (1).

The following result is less trivial but still standard.

1.4. LEMMA. (1) *Any* $t \in T(A)$ *determines uniquely a vector* $\lambda(t) = (\bar{\lambda}_0; \bar{\lambda}_1; \lambda) \in \mathbb{R}_+^{r_0} \times \mathbb{R}_+^{r_1} \times \mathbb{R}_+$ *with* $\bar{\lambda}_x = (\lambda_{x_1}, ..., \bar{\lambda}_{x_{r_x}})$ *for* $x = 0$ *or* 1 *such that*

$$\min\{\lambda_{0_i} : 1 \leqslant i \leqslant r_0\} = \min\{\lambda_{1_i} : 1 \leqslant i \leqslant r_1\} = 0, \tag{1-1}$$

and

$$t = \sum_{x \in \{0,\,1\}} \sum_{i=1}^{r_x} \lambda_{x_i} \cdot \delta_{x_i} + \lambda \cdot \mu, \tag{1-2}$$

where μ *is a Radon probability measure on* $[0, 1]$ (*cf. Lemma* 1.3), *which is determined by* t *if* $\lambda \neq 0$.

We shall call (1-2), *with condition* (1-1), *the standard form of* t. *We shall also call* μ *in* (1-2) *the principal part of* t *and the rest residual part.*

(2) *Any* $f \in \mathit{Aff}(T(A))$ *defines a real-valued Unction* $\delta^*(f)$ *on* $sp(A)$:

$$\delta^*(f)(x) = f(\delta_x).$$

The map δ^* *identifies* $\mathit{Aff}(T(A))$, *as an ordered space with unit, with the space of all real-valued function* f *on* $sp(A)$ *satisfying the following*:

(i) f *is continuous on* (0, 1);

(ii) $x = 0$ *or* 1,

$$\lim_{[\tilde{x}] \to x} f(\tilde{x}) = \sum_i \frac{n_{x_i}}{n} f(x_i),$$

where limit is taken in $[0, 1]$. (*Recall that* $[\,\cdot\,] : sp(A) \to [0, 1]$ *is the canonical quotient map.*)

Let B be a splitting interval algebra. Then by Lemma 1.4, to define a morphism $\theta : T(B) \to T(A)$, it suffices to define $\theta(\delta_y)$ for $y \in sp(B)$ in a way which is continuous on $(0, 1) \subset sp(B)$ and satisfies the right boundary condition. We will do this a few times implicitly.

To end this section, we mention a concept which will be useful for this paper.

1.5. Theorem (cf. [Su] Theorem 3.1.) *Let A, B be two splitting interval algebras. For any genital map $\phi: A \to B$, any finite set $F \in A$ and any $\varepsilon > 0$, there exist a unital map $\phi': A \to B$ such that:*

(1) $\|\phi(f) - \phi'(f)\| < \varepsilon$, *for* $f \in F$;

(2) *there exist continuous maps* $\{s_j(\cdot)\}_{j=1}^{p} \subset C([0,1]; sp(A))$ *and a unitary* $W \in M_m(C[0,1])$, *where* M_m *is the generic fibre of* B, *such that*

$$\phi'(f)(y) = W(y) \operatorname{diag}(f(s_1(y)), ..., f(s_p(y)))\, W^*(y) \tag{1-3}$$

for all $f \in A$, $y \in [0,1]$.

Any homomorphism of the form (1-3) will be called standard, and $\{s(\cdot)\}_{j=1}^{p}$ its eigenvalue maps. More generally, a homomorphism between two finite direct sums of splitting interval algebras will be called standard, if on each summand of B the map assumes the form (1-3), where $\{s_j(\cdot)\}_{j=1}^{p}$ are now continuous maps from $[0,1]$ to the spectrum of A.

2. COMPATIBLE PAIRS AND ALMOST COMPATIBLE PAIRS

This section serves a dual purpose: we shall collect a few elementary facts about compatible pairs for splitting interval algebras, thus laying the foundation for the structure theory to be developed in Section 3; And we shall prove that an almost compatible pair for splitting interval algebras is close to a compatible pair, a fact that we shall need in the early stage of our proof of the main theorem in Section 5.

Let A, B be two C^*-algebras. Recall that two morphisms $\kappa: K_0(A) \to K_0(B)$ and $\theta: T(B) \to T(A)$ are compatible if

$$\langle e, \theta(t) \rangle = \langle \kappa(e), t \rangle$$

for any $e \in K_0(A)$ and $t \in T(B)$. For brevity, we shall call $(\kappa; \theta)$ a compatible pair for $(A; B)$.

2.1. Definition. Let $A, B_1, ..., B_p$ be C^*-algebras and let $B = B_1 \oplus \cdots \oplus B_p$. For morphisms $\kappa_j: K_0(A) \to K_0(B_j)$ and $\theta_j: T(B_j) \to T(A)$, define $\kappa: K_0(A) \to K_0(B)$ by

$$\kappa = \kappa_1 \oplus \cdots \oplus \kappa_p \tag{2-1}$$

and $\theta: T(B) \to T(A)$ by

$$\theta(t) = \sum_j \theta_j(t_j) \tag{2-2}$$

for any $t \in T(B)$, where t_j is the restriction of t on B_j, and θ_j is extended by $\theta_j(\lambda \cdot t) \overset{\text{def}}{=} \lambda \cdot \theta_j$ for any $\lambda \geqslant 0$ and $t \in T(B_j)$. $(\kappa; \theta)$ will be denoted by $\sum_j^{\oplus} (\kappa_j; \theta_j)$.

2.2. Lemma. *(1) $\sum_j^{\oplus} (\kappa_j; \theta_j)$ is a compatible pair for $(A; B)$, if and only if each $(\kappa_j; \theta_j)$ is compatible for $(A; B_j)$.*

(2) For any compatible pair $(\kappa; \theta)$ for $(A; B)$, where $B = B_1 \oplus \cdots \oplus B_p$, there exists a (unique) compatible pair $(\kappa_j; \theta_j)$ for $(A; B_j)$ such that: $(\kappa; \theta) = \sum_j^{\oplus} (\kappa_j; \theta_j)$.

Proof. Part (1) is straightforward. To prove part (2) let $Q_j\colon B \to B_j$ be the canonical quotient map. Then $(\kappa_j; \theta_j) = (Q_{j*} \circ \kappa; \theta \circ Q_j^*)$ will do. ∎

We shall need to analyze the structure of compatible pairs $(\kappa; \theta)$ for $(A; B)$, where A, B are finite direct sums of splitting interval algebras. By Lemma 2.2, we can focus on those where B is a splitting interval algebra. Our next step is to show that we can in fact assume that A is a splitting interval algebra, too.

Note that if $A = A_1 \oplus \cdots \oplus A_p$, then

$$T(A) = \left\{ \sum_j \lambda_j \cdot t_j \colon (\lambda_1, ..., \lambda_p) \in \Delta^p,\ t_j \in T(A_j) \right\},$$

where Δ^p is the standard p-simplex.

2.3. Lemma. *Let $A = A_1 \oplus \cdots \oplus A_p$, B a splitting interval algebra, and $(\kappa; \theta)$ a compatible pair for $(A; B)$. Suppose that $\kappa|_{A_j} \neq 0$ for each j. Then there exist C^*-algebras $B_1, ..., B_p$, an injective (unital) homomorphism $\phi\colon \bigoplus_j B_j \to B$, and a compatible pair $(\kappa_j; \theta_j)$ for each $(A_j; B_j)$, such that*

$$\kappa\left(\bigoplus_j e_j\right) = \phi_*\left(\bigoplus_j \kappa_j(e_j)\right) \tag{2-3}$$

for $\bigoplus_j e_j \in K_0(A)$ and

$$\theta(t) = \sum_j \lambda_j \cdot \theta_j(t_j) \tag{2-4}$$

for $t \in T(B)$, where $t_j \in T(B_j)$ and $\sum_j \lambda_j \cdot t_j = \phi^*(t)$.

Note that if $\kappa|_{A_j} \equiv 0$ for certain j, then $\theta(T(B)) \subset T(A \ominus A_j)$, hence $(\kappa; \theta)$ is "basically" a compatible pair for $(A \ominus A_j; B)$.

Proof. Let $B = \mathscr{S}(\bar{m}_0; \bar{m}_1)$ with

$$B(0) = \bigoplus_{i=1}^{s_0} M_{k_{0_i}}(\mathbb{C}), \qquad B(1) = \bigoplus_{i=1}^{s_1} M_{k_{1_i}}(\mathbb{C}).$$

Recall that

$$K_0(B) \cong \left\{ (\bar{k}_0; \bar{k}_1) \in \mathbb{Z}^{s_0} \times \mathbb{Z}^{s_1} : \sum_i^{s_0} k_{0_i} = \sum_i^{s_1} k_{1_i} \right\}.$$

Note that for any $e \in K_0(B)$, if $0 < e \leqslant [1_B]$, then there is a (self-adjoint) projection $0 \neq E \in B$ such that $[E] = e$. Let $e_j = \kappa(1_{A_j})$ for each j. Choose orthogonal projections $E_1, ..., E_p \in B$ such that $[E_j] = e_j$ and $1_B = E_1 + \cdots + E_p$. Let $B_j = E_j B E_j$ be the "cut-down" subalgebra of B.

It is easy to show that each B_j is a splitting interval algebra. In fact, $B_j = \mathscr{S}(\kappa(1_{A_j}))$ (after throwing out entries in $\kappa(1_{A_j})$ that are 0). we shall now construct a compatible pair $(\kappa_j; \theta_j)$ for each $(A_j; B_j)$.

To construct κ_j, note that the canonical inclusion map $B_j \hookrightarrow B$ induces an identification of $K_0(B_j)$ with a subgroup of $K_0(B)$: if $e_j = (\bar{k}_0(j); \bar{k}_1(j))$, then

$$K_0(B_j) = \{ (\bar{k}_0; \bar{k}_1) \in K_0(B) : k_{x_i} = 0 \text{ if } k_{x_i}(j) = 0 \},$$

where $e_j = \kappa(1_{A_j}) = (k_{0_1}(j), ..., k_{0_{s_0}}, k_{1_1}(j), ..., k_{1_{s1}}(j))$. Therefore, $\kappa(K_0(A_j)) \subseteq K_0(B_j)$. And we define κ_j to be the restriction of κ on $K_0(A_j)$.

To define θ_j, let $k_y(j) = \langle \kappa(1_{A_j}), \delta_y \rangle$ for any $y \in sp(B)$. Then $sp(B_j)$ can be identified with a subset of $sp(B)$, as follows:

$$sp(B_j) = \{ y \in sp(B) : k_y(j) \neq 0 \}.$$

We then define, for any $y \in sp(B_j)$,

$$\theta_j(\delta_y) = \frac{1}{k_y(j)} \theta(\delta_y) \bigg|_{A_j},$$

where the δ_y on the left hand side is a tracial state on B_i while the δ_y on the right hand side a tracial state on B. It is straightforward to check that θ_j extends to a morphism from $T(B_j)$ to $T(A_j)$ which is compatible with κ_j.

Finally, let ϕ: $\bigoplus_j B_j = B_1 + \cdots + B_p \to B$ be the natural inclusion map. Then it is easy to check (2-3) and (2-4). ∎

Therefore, we can now focus on compatible pairs for splitting interval algebras (without worrying about direct sums).

2.4. Lemma. *Let A, B be two splitting interval algebras. Two morphisms $\kappa: K_0(A) \to K_0(B)$ and $\theta: T(B) \to T(A)$ are compatible if, and only if:*

(1) *$\lambda(\theta(\delta_y))$ is independent of $y \in [0, 1]$ with λ being defined in* 1.4; *and*

(2) *$((Q_y)_* \cdot \kappa; \theta \cdot (Q_y)^*)$ is compatible for $y = 0$ or 1, where $Q_y: B \to B(y)$ is the canonical evaluation map at y.*

Proof. Direct computations. ∎

Next we shall study compatible pairs for $(A; B)$, where A is a splitting algebra and $B = M_m(\mathbb{C})$. Note that $T(B)$ consists of only one element, namely, the normalized trace Tr on B. Therefore, a morphism $\theta: T(B) \to T(A)$ can be identified with the tracial state $\theta(\mathrm{Tr}) \in T(A)$. Recall that a tracial state on a C^*-algebra naturally induces a map from the K_0 group of the algebra to $\mathbb{R}$.

2.5 Lemma. (1) *Two traces $t_1, t_2 \in T(A)$ induces the same map on $K_0(A)$ if and only if $\lambda(t_1) = \lambda(t_2)$ (cf. Definition* 1.4).

(2) *A pair of morphisms $(\kappa; \theta)$ for $(A; B)$ is compatible, if and only if, $\kappa = m \cdot \theta(\mathrm{Tr})_*$, where $\theta(\mathrm{Tr})_*$ is the map on $K_0(A)$ induced by the trace $\theta(\mathrm{Tr}) \in T(A)$.*

(3) *If $(\kappa; \theta)$ is a compatible pair for $(A; B)$, then κ determines $\lambda(\theta(\mathrm{Tr}))$ and vise versa.*

Proof. For $1 \leqslant i \leqslant r_0$ and $1 \leqslant j \leqslant r_1$, let $e_{i, j}$ be a rank one projection in A satisfying:

$$e_{i, j}(0) \in A(0_i), \qquad \text{and} \qquad e_{i, j}(1) \in A(1_j).$$

Note that $[e_{i, j}] \in K_0(A)$ is independent of the specific choice of this projection. Let $G = \{[e_{i, j}]: 1 \leqslant i \leqslant r_0, 1 \leqslant j \leqslant r_1\}$. This is a generating set for (the positive cone of) $K_0(A)$.

Now for any trace $t \in T(A)$, let

$$t = \sum_{x \in \{0, 1\}} \sum_{k=1}^{r_x} \lambda_{x_k} \delta_{x_k} + \lambda \cdot \mu$$

be its standard form. It induces a map $t_*: K_0(A) \to \mathbb{R}$. Choose $e_{i_0, j_0} \in G$ such that $t_*(e) = \min\{t_*(g): g \in G\}$, i.e., $\lambda_{0_{i_0}} = \lambda_{1_{j_0}} = 0$. Then it is easy to check that

$$\lambda = n \cdot t_*(e_{i_0, j_0}), \qquad \lambda_{0_k} = n_{0_k} \cdot (t_*(e_{k, j_0}) - t_*(e_{i_0, j_0})),$$

and similarly

$$\lambda_{1_k} = n_{1_k} \cdot (t_*(e_{i_0,\,k}) - t_*(e_{i_0,\,j_0})).$$

The lemma follows. ∎

These lemmas are the basics for a structure theory of compatible pair for splitting interval algebras, which will be developed in the next section. In the rest of this section, we shall discuss almost compatible pairs.

Let $A = A_1 \oplus \cdots \oplus A_p$, where each A_j is a splitting interval algebra, and let $G = G_1 \cup \cdots \cup G_p$, where each G_j is the generating set for $K_0(A_j)$ as being described in the proof of Lemma 2.5. Let B be a C^*-algebra and $\varepsilon > 0$. By a ε-compatible pair $(\kappa; \theta)$ for $(A; B)$, we shall mean two morphisms $\kappa\colon K_0(A) \to K_0(B)$ and $\theta\colon T(B) \to T(A)$ satisfying the condition

$$|\langle e, \theta(t)\rangle - \langle \kappa(e), t\rangle| < \varepsilon$$

for all $e \in G$ and $t \in T(B)$.

2.6. Lemma. *Let A be a splitting interval algebra whose generic fibre is $M_n(\mathbb{C})$. Let $B = M_m(\mathbb{C})$. Then for any ε-compatible pair $(\kappa; \theta^\varepsilon)$ for $(A; B)$, there exists a compatible pair $(\kappa; \theta)$ for $(A; B)$ such that*

$$\|\theta^\varepsilon(\mathrm{Tr}) - \theta(\mathrm{Tr})\| < 9n^2\varepsilon, \tag{2-5}$$

where Tr *is the normalized trace on B, and $\|\cdot\|$ is the norm on A^*.*

Proof. Suppose that

$$t^\varepsilon = \sum_{x \in \{0,1\}} \sum_{k=1}^{r_x} \lambda^\varepsilon_{x_k} \delta_{x_k} + \lambda^\varepsilon \cdot \mu^\varepsilon$$

is the standard form of $t^\varepsilon = \theta_\varepsilon(\mathrm{Tr})$. Let

$$t = \sum_{x \in \{0,1\}} \sum_{k=1}^{r_x} \lambda_{x_k} \delta_{x_k} + \lambda \cdot \mu^\varepsilon,$$

where λ, λ_{x_k} are as calculated in the proof of Lemma 2.5 with t_* there replaced by κ/m. Note that κ is a map from $K_0(A)$ to $\mathbb{Z} \subset \mathbb{R}$. By Lemma 2.5, the morphism $\theta\colon T(B) \to T(A)$ defined by $\theta(\mathrm{Tr}) = t$ is compatible with κ. Moreover, the ε-compatibility condition for $(\kappa; \theta^\varepsilon)$ now translates into the following:

$$|\langle e, t^\varepsilon\rangle - \langle e, t\rangle| < \varepsilon.$$

With the same notations as in the proof of Lemma 2.5, we have

$$|\lambda^{\varepsilon} - \lambda| < n \cdot \varepsilon,$$
$$|\lambda^{\varepsilon}_{0_{i_0}} - \lambda_{0_{i_0}}| < 2n_{0_{i_0}}\varepsilon,$$
$$|\lambda^{\varepsilon}_{1_{j_0}} - \lambda_{1_{j_0}}| < 2n_{1_{j_0}}\varepsilon,$$

and

$$|\lambda^{\varepsilon}_{x_k} - \lambda_{x_k}| < 4n_{x_k}\varepsilon.$$

It then follows that

$$\|\theta^{\varepsilon}(\mathrm{Tr}) - \theta(\mathrm{Tr})\| < 9n^2\varepsilon. \quad \blacksquare$$

It is important to note that the right hand side of (2-5) depends only upon ε and the size of A, and does not depend on the second algebra.

The following proposition can be proved in the same way, with the help of Lemmas 2.2, 2.3, and 2.4.

2.7. PROPOSITION. *Let A be a finite direct sum of splitting interval algebras. Then there is a constant $c(A)$ such that for any ε-compatible pair $(\kappa; \theta^{\varepsilon})$ for $(A; B)$, where B is also a finite direct sum of splitting interval algebras, there exist a compatible pair $(\kappa; \theta)$ for $(A; B)$ such that*

$$\|\theta^{\varepsilon}(t) - \theta(t)\| < c(A) \cdot \varepsilon$$

for all $t \in T(B)$, where $\|\cdot\|$ is the norm on A^*.

For example, if $A = \bigoplus_{j=1}^{p} A_j$ and the generic fibre of A_j is $M_{n_j}(\mathbb{C})$, then we can take

$$c(A) = \sum_{j=1}^{p} n_j(9n_j^2 + 9n_j + 1).$$

3. LOCAL EXISTENCE

In this section we shall establish a local existence result which will be used in the proof of the main theorem in Section 5. Our approach hinges heavily upon an analysis of the algebraic structure of compatible morphisms on K_0 group and on tracial state spaces of splitting interval algebras. It turns out that compatible pairs between splitting interval algebras "break up into the sum of some basic, more manageable pairs." To make this more precise, we introduce the following definitions:

3.1 DEFINITION. Let (κ, θ) be a compatible pair for $(A; B)$. A decomposition of (κ, θ), denoted $(\kappa; \theta) = \sum_j (\kappa_j; \theta_j)$, consists of

(1) mutually orthogonal C^*-subalgebras $B_1, B_2, ..., B_p$ of B such that $1_B \in B_1 + \cdots + B_p$;

(2) a compatible pair $(\kappa_j; \theta_j)$ for (A, B_j) for each j; that satisfy

$$\kappa = \iota_* \left(\bigoplus_j \kappa_j \right),$$

where $\iota\colon \bigoplus_j b_j = B_1 + \cdots + B_p \hookrightarrow B$ is the inclusion map, and

$$\theta(t) = \sum_j \theta_j(t\,|_{B_j}), \qquad t \in T(B),$$

where θ_j is again naturally extended.

This definition should be compared with Definition 2.1 and the constructions in Lemma 2.3.

3.2. DEFINITION. Let (κ_i, θ_i) be a compatible pair for $(A_i; B_i)$, $i = 1, 2$. (κ_1, θ_1) is equivalent to (κ_2, θ_2), if there are isomorphisms $\phi\colon A_1 \to A_2$ and $\psi\colon B_1 \to B_2$ such that $\kappa_2 = \psi_* \kappa_1 \phi_*^{-1}$ and $\theta_2 = (\phi^{-1})^* \, \theta_1 \psi^*$.

Here are some basic compatible pairs that we will encounter:

3.3. DEFINITION. Let $A = \mathscr{S}(\bar{n}_0; \bar{n}_1)$ be a splitting interval algebra. Recall that $K_0(A) = \{(\bar{k}_0; \bar{k}_1) \in \mathbb{Z}^{r_0} \times \mathbb{Z}^{r_1} : \sum_i k_{0_i} = \sum_i k_{1_i}\}$.

(1) Let $B = M_n(\mathbb{C})$, $\kappa(\bar{k}_0; \bar{k}_1) = \sum_i k_{0_i}$ for $(\bar{k}_0; \bar{k}_1) \in K_0(A)$ and $\theta(\mathrm{Tr}) = \mu$ where Tr is normalized trace on B and μ is any Radon probability measure on $[0, 1]$ (cf. Lemma 1.3). Then $(\kappa; \theta)$ is compatible for $(A; B)$. Any compatible pair equivalent to such a pair will be called *generic*.

(2) Let $x \in \{0, 1\}$ be a broken endpoint. Let $B = A(x)$, $\kappa(\bar{k}_0; \bar{k}_1) = \bar{k}_x$ for $(\bar{k}_0; \bar{k}_1) \in K_0(A)$ and $\theta(\delta_{x_i}) = \delta_{x_i}$, for $1 \leqslant i \leqslant r_x$. A compatible pair equivalent to this will be called *broken* (at x).

(3) Let x_i be a fractional endpoint of A. And let $B = A(x_i)$, $\kappa(\bar{k}_0; \bar{k}_1) = k_{x_i}$ for $(\bar{k}_0; \bar{k}_1) \in K_0(A)$ and $\theta(\mathrm{Tr}) = \delta_{x_j}$ where Tr is the normalized trace on B. A compatible pair equivalent to this will be called *fractional* (at x_i).

3.4. *Remark.* It is easy to see that for a compatible pair $(\kappa; \theta)$ for $(A; B)$, it is generic if and only if B is isomorphic to the generic fibre of A and κ is faithful; And it is broken (or fractional) at x if and only if it is equivalent to the compatible pair induced by the canonical evaluation map $Q_x\colon A \to A(x)$. We also note that a broken pair has a natural decomposition by the corresponding fractional pairs. Finally, note that for any finite dimensional C^*-algebra B, there is a unique normalized trace Tr on B. In all of these three cases of compatible pairs in Definition 3.3, $\theta(\mathrm{Tr})$ is principal if the pair is either generic or broken, and is residual if the pair if fractional (cf. Lemma 1.4).

The proof of the local existence result in this section (Theorem 3.7) has two main ingredients: a modified version of an approximation theorem by Thomsen [T1] as improved by Li [Li], and a reduction theory for compatible pairs. The following lemma will be basic for the reduction process.

3.5. Lemma. *Let A be a splitting interval algebra, $B=\bigoplus_s M_{m_s}(c)$, $m=\sum_s m_s$ and $(\kappa;\theta)$ a compatible pair for $(A;B)$. Then there exists a decomposition $(\kappa;\theta)=\sum_j(\kappa_j;\theta_j)$ such that*

(1) *each $(\kappa_j;\theta_j)$ is either generic, broken or fractional; and*

(2) *if $\theta(\mathrm{Tr})=\sum_{x\in\{0,1\}}\sum_{i=1}^{r_x}\lambda_{x_i}\cdot\delta_{x_i}+\lambda\cdot\mu$ be the standard form for $\theta(\mathrm{Tr})\in T(A)$ (cf. Lemma 1.4), where* Tr *is the special normalized trace on B, then the number of fractional pairs at x_i is exactly $m\cdot\lambda_{x_i}/n_{x_i}$.*

Proof. First we treat the special case where $B=M_m(\mathbb{C})$. Note that in this case, $K_0(B)=\mathbb{Z}$ and $T(B)=\{\mathrm{Tr}\}$, where Tr is the normalized trace on B. Let

$$\theta(\mathrm{Tr})=\sum_{x\in\{0,1\}}\sum_{i=1}^{r_x}\lambda_{x_i}\cdot\delta_{x_i}+\lambda\cdot\mu$$

be the standard from for $\theta(\mathrm{Tr})\in T(A)$.

Now suppose $A=\mathscr{S}(\bar{n}_1;\bar{n}_1)$. Recall that

$$K_0(A)=\left\{(\bar{k}_0;\bar{k}_1)\in\mathbb{Z}^{r_0}\times\mathbb{Z}^{r_1}:\sum_i k_{0_i}=\sum_i k_{1_i}\right\}.$$

Compatibility condition implies that

$$\kappa(\bar{k}_0;\bar{k}_1)=\sum_{x\in\{0,1\}}\sum_{i=1}^{r_x}\frac{m\cdot\lambda_{x_i}}{n_{x_i}}k_{x_i}+\frac{m\cdot\lambda}{n}\sum_i k_{0_i}$$

for $(\bar{k}_0;\bar{k}_1)\in K_0(A)$. It follows that $m\cdot\lambda/n$, $m\cdot\lambda_{x_i}/n_{x_i}\in\mathbb{N}$. Note that

$$\theta_j(\mathrm{Tr}|_{B_j})=\frac{n_{x_i}}{m}\delta_{x_i}$$

if $(\kappa_j; \theta_j)$ is fractional at x_i, and there is no contribution to the residual part of $\theta(\mathrm{Tr})$ from generic pairs. The special case then follows immediately. Note that there is no broken pair in this decomposition.

The general case follows by applying the special case to each of the simple blocks of B. Observe that, in the general case, some fractional pairs arising from different blocks of B might add up into a broken pair, and we must consolidate them into a single (broken) pair by taking direct sum whenever this happens. Again, note that

$$\theta(\mathrm{Tr}) = \sum_j \theta(\mathrm{Tr}|_{B_j}),$$

and only fractional pairs contribute to the residual part of $\theta(\mathrm{Tr})$,

$$\theta_j(\mathrm{Tr}|_{B_j}) = \frac{n_{x_i}}{m}\,\delta_{x_i}$$

if $(\kappa_j; \theta_j)$ is fractional at x_i. This completes the proof. ∎

The following lemma is the approximation result that we alluded to. It is essentially due to Thomsen [T1] and Li [Li].

3.6. THEOREM. *For any finite set $F \in C[0,1]$ and any $\varepsilon > 0$, there exists a constant N such that for any morphism $\theta: T(C[0,1]) \to T(C[0,1])$ and any $q > N$, there are exactly q endomorphisms ϕ_k of $C[0,1]$ satisfying*

$$\left\| \theta(t)(f) - \frac{1}{q} \sum_k \phi_k^*(t)(f) \right\| < \varepsilon, \tag{2-4}$$

for any $f \in F$, $t \in T(C[0,1])$. Moreover, if $\theta(\delta_0) = 1/n \sum_{j=1}^n \delta_{x(j)}$ for some $x(j) \in [0,1]$ and/or $\theta(\delta_1) = 1/n \sum_{j=1}^n \delta_{y(j)}$ for some $y(j) \in [0,1]$, and $n \mid q$, then those q endomorphisms can be chosen such that $\theta(\delta_0) = 1/q \sum_{k=1}^q \phi_k^*(\delta_0)$ and/or $\theta(\delta_1) = 1/q \sum_{k=1}^q \phi_k^*(\delta_0)$.

Proof. The first part is a special case of a Theorem by Li [Li]. Thomsen [T1] first establishes the existence of endomorphisms satisfying (2-4). Li [Li] improves his result by pointing out that the number N depends only on F and ε and any number $q \geqslant N$ will do. Their proofs can be modified to prove the second part of this lemma. ∎

The following is the main result of this section:

3.7. THEOREM. *Let A be a splitting interval algebra. Then for any finite set $F \in A$ and any $\varepsilon > 0$, there is a constant $N \in \mathbb{N}$, such that for any compatible pair $(\kappa; \theta)$ for $(A; B)$, where B is also a splitting interval algebra,* *there is a homomorphism $\phi: A \to B$ of the standard form* (*cf.* Section 1) *which induces κ and almost induces θ in the sense that*

$$\|\phi^*(t)(f)-\theta(f)\| < \varepsilon + \frac{2+r_0+r_1}{m} \cdot N \cdot \|f\|, \tag{3-1}$$

for any $f \in F$, $t \in T(B)$.

Proof. Let $\widetilde{F} = \{\mathrm{Tr}(f) \in C[0,1] : f \in F\}$ where $\mathrm{Tr}(f)$ is the function given by taking the trace of f at each point. Let $N = N(\widetilde{F}; \varepsilon/2)$ as in Lemma 2.7. By Lemma 2.7, we can assume that $n \mid N$, where n is the size of the generic fibre of A.

We will assume that there is $\delta > 0$ such that $\theta(\delta_y)$ remains constant for $y \in (0, \delta)$ and for $y \in (1-\delta, 1)$. If θ does not satisfy this condition, we can find, by a small perturbation around the boundary, some θ' which is compatible with κ, satisfies this condition, and is close to θ in that

$$\|\theta(t)(f) - \theta'(t)(f)\| < \varepsilon/2$$

for all $f \in F$, $t \in T(B)$. To simplify the notation, we simply assume that θ satisfies the condition. And the next step is to show that such a pair allows a decomposition analogous to the one we have in Lemma 3.5.

In fact, the decomposition we need follows basically from Lemma 3.5. The idea is to use Lemma 3.5 to decompose the "boundaries" of the pair, and then "fill in the interior." To be more precise, for $x = 0$ or 1, let $Q_x: B \to B(x)$ be the canonical evaluation map. Applying Lemma 3.5 to the compatible pairs $(\kappa^{(x)}; \theta^{(x)}) \stackrel{\text{def}}{=} ((Q_x)_* \circ \kappa; \theta \circ (Q_x)^*)$, we get decompositions $(\kappa^{(x)}; \theta^{(x)}) = \sum_j (\kappa_j^{(x)}; \theta_j^{(x)})$.

Note that by Lemmas 2.4(1) and 3.5(2), these two decompositions have the same total number of summand pairs and the same number of summand pairs fractional at x_i for each fractional endpoint x_i. We therefore assume that $(\kappa_j^{(0)}; \theta_j^{(0)})$ is fractional at x_i if and only if $(\kappa_j^{(1)}; \theta_j^{(1)})$ is fractional at the same point x_i.

We have more alignment to do. We will group the rest compatible pairs into "batches," in the following way: In each simple block of $B(0)$, we group the resulting *generic* pairs into batches of N pairs. There are at most $N-1$ pairs which remain unaligned in each block of $B(0)$. Then we consider broken pairs in the decomposition of $(\kappa^{(0)}; \theta^{(0)})$, and again group them into batches of N *equivalent* broken pairs (that is, pairs in the same batch are broken at the same point). There are at most $2(N-1)$ broken pairs that remain unaligned. Overall, the total number of unaligned pairs in the decomposition of $(\kappa^{(0)}; \theta^{(0)})$ will not exceed $(2+s_0)(N-1)$, where s_0 is the number of simple blocks of $B(0)$. Similarly, we align compatible pairs in the decomposition of $(\kappa^{(1)}; \theta^{(1)})$. To fix the notation, let us assume that the total number of batches at 0 does not exceed that at 1, and that if two pairs $(\kappa_j^{(0)}; \theta_j^{(0)})$ and $(\kappa_l^{(0)}; \theta_l^{(0)})$ are in the same batch, so are $(\kappa_j^{(1)}; \theta_j^{(1)})$ and $(\kappa_l^{(1)}; \theta_l^{(1)})$.

We now "fill in the interior" to get a decomposition of $(\kappa; \theta)$. On the algebra level, it is easy to find mutually orthogonal subalgebras $B_1, ..., B_p$ of B such that:

(1) if $(\kappa_j^{(0)}; \theta_j^{(0)})$ is fractional, then $B_j \cong B_j^{(0)}$;

(2) if $(\kappa_j^{(0)}; \theta_j^{(0)})$ is not fractional, B_j is a splitting interval algebra with $B_j(0) = B_j^{(0)}$ and $B_j(1) = B_j^{(1)}$.

On the K_0 level, if B_j is finite dimensional (as in case (1)), we define $\kappa_j = \kappa_j^{(0)}$; and if B_j is a splitting interval algebra, we define $\kappa_j = \kappa_j^{(0)} \oplus \kappa_j^{(1)}$.

To define the morphism between tracial state spaces, let

$$\theta(\delta_y) = \sum_{x \in \{0, 1\}} \sum_{i=1}^{r_x} \lambda_{x_i} \delta_{x_i} + \lambda \mu(y)$$

be the standard form of $\theta(\delta_y)$ (by Lemma 2.4(2), λ_{x_k} and λ are independent of $y \in [0, 1]$). If B_j is finite dimensional, we define $\theta_j = \theta_j^{(0)}$; if B_j is a splitting interval algebra, we define, for any $y \in sp(B_j)$,

$$\theta_j(\delta_y) = \begin{cases} \theta_j^{(0)}(\delta_y), & [y] = 0, \\ \left(1 - \frac{t}{\delta}\right) \theta_j^{(0)}(\delta_0) + \frac{t}{\delta} \mu(y), & [y] \in \left(0, \frac{t}{\delta}\right), \\ \mu(y) & [y] \in \left[\frac{t}{\delta}, 1 - \frac{t}{\delta}\right], \\ \frac{t}{1-\delta} \mu(y) + \left(1 + \frac{t}{1-\delta}\right) \theta_j^{(1)}(\delta_1), & [y] \in \left(1 - \frac{t}{\delta}, 1\right), \\ \theta_j^{(1)}(\delta_y), & [y] = 1, \end{cases}$$

where $[\cdot]: sp(B_j) \to [0, 1]$ is the canonical quotient map.

By construction $\theta_j: sp(B_j) \to T(A)$ is continuous on $(0, 1)$ and satisfies the right boundary conditions, and thus defines a morphism $\theta_j: T(B_j) \to T(A)$. And it is easy to check that $(\kappa_j; \theta_j)$ is compatible.

This completes a decomposition of $(\kappa; \theta)$. It remains to be seen that we can change the θ_j's defined above to get a homomorphism from A to B. And this is where the batching becomes necessary.

Note that each fractional pair can be realized by a homomorphism (namely, the canonical evaluation map).

Let us agree to say that two pairs $(\kappa_j; \theta_j)$ and $(\kappa_l; \theta_l)$ are in the same batch if $(\kappa_j^{(0)}; \theta_j^{(0)})$ and $(\kappa_l^{(0)}; \theta_l^{(0)})$ are in the same batch. Note that by construction, pairs in the same batch are equivalent. By Theorem 3.6, in any batch, each compatible pair can be replaced by one induced by a C^*-homomorphism in a way which does not change the K_0-map and approximates the average of the tracial state maps.

Finally, for those unaligned pairs, we take any *injective* homomorphism ϕ_j which induces the right map on K_0 group and also induces the right morphism on tracial state space *on the boundary*. This can always be done since the fibre of B_j at the boundary is a fibre of A. Then we replace the compatible pair by the one induced by this homomorphism. We do not have much control about how closed this new one is to the original one, but fortunately such pairs are relatively few. ▮

3.8. *Remark.* (1) As we shall see later in Section 5, in an inductive sequence of finite direct sums of splitting interval algebras with simple inductive limit, the size of each block of any summand will have to explode, hence, the quotient on the right hand side of (3-1) will become as small as we want.

(2) In general, the homomorphism ϕ might fail to be injective. But if in the decomposition of $(\kappa_j^{(0)}; \theta_j^{(0)})$ (or, equivalently, of $(\kappa_j^{(1)}; \theta_j^{(1)})$, there is at least one generic or broken pair, then ϕ can be chosen to be injective. See the last part of the proof. Again, it follows from Lemma 5.2 that eventually there will be many generic pairs in the decomposition of each simple block of any direct summand of the inductive sequence with simple limit.

4. LOCAL UNIQUENESS

Recall (from [E1]) that for given $\varepsilon > 0$ two unital *-homomorphisms $\phi_0, \phi_1 \colon A \to B$ are said to be approximately unitarily equivalent on a given finite subset $F \subset A$ to within ε, if there is a unitary u in B such that

$$\|u\phi_0(f)\, u^* - \phi_1(f)\| < \varepsilon, \qquad f \in F.$$

In this section we shall determine when two unital $*$-homomorphisms between two finite direct sums of splitting algebras are approximately unitarily equivalent in the above sense.

The following proposition can be found in [Su], Prop. 7.3:

4.1. Proposition. *Let $\phi, \psi \colon A \to B$ be two unital standard maps between two splitting interval algebras. Suppose that:*

(1) $\phi_* = \psi_* \colon K_0(A) \to K_0(B)$; *and*

(2) *ϕ and ψ have the same eigenvalue maps.*

Then for any finite set $F \in A$ and any $\varepsilon > 0$, there is a unitary $u \in B$ such that

$$\|u\phi(f)\, u^* - \psi(f)\| < \varepsilon$$

for all $f \in F$.

We now prove the main theorem of this section.

4.2. THEOREM. *Let A_1 and A_2 be two finite direct sums of splitting interval algebras, let ϕ and ψ be two unital homomorphisms from A_1 to A_2, let $F \subset A_1$ be a finite subset, let $n > 0$ be an integer and let $\varepsilon > 0$. Suppose that:*

(1) $\phi_* = \psi_*$,

(2) *for some $\delta > 0$, the images, under each of ϕ and ψ, of the canonical central generators of each minimal direct summand of A_1 (i.e., the, function $h(t) = t$) in each primitive quotient of A_2 has at least the fraction δ of its eigenvalues in each of the n consecutive subintervals of $[0, 1]$ of length $1/n$,*

(3) *the maps from TA_2 to TA_1 induced by ϕ and ψ agree to strictly within δ on the n central elements of each minimal direct summand of A_1 corresponding to the functions which are equal to zero from 0 to r/n, equal to one from $(r+1)/n$ to 1, and linear in between, $r = 0, 1, ..., n-1$, and*

(4) *for $f \in F$ and $x_1, x_2 \in [0, 1]$ with distance within $3/n$,*

$$\|f(x_1) - f(x_2)\| < \varepsilon/2.$$

It follows that there exists a unitary $u \in A_2$ such that

$$\|\phi(f) - Ad(u)\psi(f)\| < \varepsilon, \qquad f \in F.$$

Proof. It is clear that we can assume that A_2 be a single splitting algebra by passing to the quotient. We now reduce A_1 to a single block. Since $\phi_* = \psi_*$, it is standard to show that there exists a unitary $X \in A_2$ such that

$$\phi(p) = X\psi(p)\, X^*$$

for all central projections $p \in A_1$ (this may require to perturb both ϕ and ψ a little and we are free to do so). Hence, we may assume that ϕ and ψ agree on the central projections of A_1 to begin with. Write $A_1 = A_1^{(1)} \oplus A_1^{(2)}$. Let P be the corresponding central projection for $A_1^{(1)}$ and denote $Q = \phi(P) = \psi(P)$.

Consider two algebras $A_1^{(1)}$ and QA_2Q. $Q\phi Q$ and $Q\psi Q$ are two unital $*$-homomorphisms from $A_1^{(1)}$ to QA_2Q. It is obvious (and can be computed directly) that $Q\phi Q$ and $Q\psi Q$ satisfy all the conditions of the theorem.

Continuing this way, notice that A_1 has only finitely many summands, we may assume that A_1 is already a single block.

By Lemma 1.5, we may suppose that ϕ and ψ have the expressions

$$\phi(f)(t) = U(t)\begin{pmatrix} f(\alpha_1(t)) & & \\ & \ddots & \\ & & f(\alpha_p(t)) \end{pmatrix} U^*(t)$$

$$\psi(f)(t) = V(t)\begin{pmatrix} f(\beta_1(t)) & & \\ & \ddots & \\ & & f(\beta_q(t)) \end{pmatrix} V^*(t),$$

where $\{\alpha_i\}_{i=1}^p$ and $\{\beta_j\}_{j=1}^q$, the so called eigenvalue maps, are in $C([0, 1], sp(A_1))$ and U and V are in $M_m(C[0, 1])$. Here M_m is the generic fibre of A_2. By Theorem 4 of [CE], we may even assume that the eigenvalue maps are distinct on $(0, 1)$.

Our first goal is to show that $p = q$, after we make necessary groupings. Since A_1 is a splitting algebra, some of $\{\alpha_i, \beta_j\}$ might be constant maps taking values at some of the small endpoints. We group them as follows. Let $\{0_i\}_{i=1}^{r_0}$ and $\{1_i\}_{i=1}^{r_1}$ be the end points of the spectrum of A_1 at $x = 0$ and $x = 1$, respectively. If $\alpha_{n_i}(t) = 0_i$ for $i = 1, ..., r_0$ then we shall group these r_0 maps into one eigenvalue map, say, $\theta(t) = 0$. We shall also deform $\theta(t)$ into a new map $\theta'(t)$ so that it is not 0 but close to 0 for, at least, $t \in (0, 1)$. If $\theta(0)$ sits in one block of A_2 at 0 or 1, we shall further deform it to a point in $(0, 1)$. This also applies to $\theta(1)$. We do this to all possible small eigenvalue maps of ϕ and ψ. The numbers p and q might be changed. We shall call the new numbers p and q again. We shall show that they are now equal. This can be achieved by looking at the K_0-maps as follows.

Write $A_1 = \mathscr{S}(\bar{n}_0; \bar{n}_1)$ and $A_2 = \mathscr{S}(\bar{m}_1; \bar{m}_1)$ where $\bar{n}_x = (n_{x,1}, ..., n_{x,r_x})$ and $\bar{m}_x = (m_{x,1}, ..., m_{x,s_x})$, $x = 0, 1$. For each α_i, if $\alpha_i(0)$ is sitting in a block of A_2 at the end 0, we deform it to 0, and if $\alpha_i(1)$ is sitting in a block of A_2 at the end 1, we deform it to 1. Do the same to β_j. After the all possible deformations, we obtain two new unital $*$-homomorphisms, say, ϕ' and ψ'. We have

$$\phi_* = \phi'_*, \qquad \psi_* = \psi'_*.$$

Set

$$I = \{f \in A_1 \mid f(0) = f(1) = 0\},$$

$$J = \{g \in A_2 \mid g(0) = g(1) = 0\}.$$

I and J are ideals of A_1 and A_2, respectively. Notice that ϕ' and ψ' map I to J. They hence also map A_1/I to A_2/J. Now the six-term exact sequence gives the following commuting diagram:

$$\begin{tikzcd}
0 \arrow[r] & K_0(A_1) \arrow[r] \arrow[d] & K_0(A_1/I) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & K_1(I) \arrow[r] & 0 \\
0 \arrow[r] & K_0(A_2) \arrow[r] & K_0(A_2/J) \arrow[r] & K_1(J) \arrow[r] & 0.
\end{tikzcd}$$

The two maps from $K_0(A_1/I)$ to $K_0(A_2/J)$ are the ones induced by ϕ' and ψ', respectively. They agree on $K_0(A_1)$. Write the difference of these two maps as

$$\eta = \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix},$$

where, for example, η_{12} is a map from the K_0 of the summand of A/I corresponding to the end 1 to the K_0 of the summand of A_2/J corresponding to the end 0. They all have integer entries. Since η vanishes on $K_0(A_1)$, we have that for the ith row of η it has the form

$$(c_i, ..., c_i; -c_i, ..., -c_i) \qquad c_i \in \mathbb{Z},$$

where the first half part with c_i belongs to η_{11} or η_{21} and the second half with $-c_i$ belongs to η_{12} or η_{22}. This computation says that at each block of A_2 at 0 or 1, the eigenvalues of ϕ' and ψ' differ by those that can be grouped into whole end points of the spectrum of A_1. Hence, the numbers of the reminder of each small end point in each such block are the same for ϕ' and ψ'. Hence the total number of such remainders for each small end point are the same for ϕ' and ψ'. These numbers equal to the corresponding ones of ϕ and ψ. This conforms that the small eigenvalue maps are the same for ϕ and ψ. Now it is clear that $p = q$.

Next, we show that $\{\alpha_i(t)\}_{i=1}^{p}$ and $\{\beta_j(t)\}_{j=1}^{p}$ can be paired to within $3/n$ one by one. The proof uses an argument of [E1]. Let h be the selfadjoint element of the centre of A_1 that was referred to in the theorem. We shall show that the eigenvalues of $\phi(h)$ and $\psi(h)$ are within $3/n$ one by one in increasing order.

Denote by k_r the characteristic function of the interval $[r/n, 1]$, $r = 1, ..., n-1$, so that

$$h_r k_{r+1} = k_{r+1}, \qquad k_r h_r = h_r, \qquad r = 1, ..., n-1,$$

where $h_1, ..., h_n$ denote the n functions specified in the statement of the theorem. By applying the normalised trace Tr of M_m, we have

$$\operatorname{Tr} \phi(k_r) \leqslant \operatorname{Tr} \phi(h_{r-1}) + \delta \leqslant \operatorname{Tr} \psi(h_{r-1}) + \delta \leqslant \operatorname{Tr} \psi(k_{r-1}) + \delta,$$

$$r = 1, ..., n-1.$$

This gives

$$\operatorname{rank} \phi(k_r) \leqslant \operatorname{rank} \psi(k_{r-1}) + m\delta.$$

Similarly, we have

$$\operatorname{rank} \psi(k_r) \leqslant \operatorname{rank} \phi(k_{r-1}) + m\delta.$$

By hypothesis, there are at least $m\delta$ eigenvalues of each of $\phi(h)$ and $\psi(h)$ in each interval of $[(r-1)/n, r/n]$. This gives us

$$\operatorname{rank} \phi(k_r) \leqslant \operatorname{rank} \psi(k_{r-2})$$

$$\operatorname{rank} \psi(k_r) \leqslant \operatorname{rank} \phi(k_{r-2})$$

for $r = 3, ..., n-1$. To see that the eigenvalues of $\phi(h)(t)$ and $\psi(h)(t)$ can be paired to within $3/n$ one by one, suppose that there are l eigenvalues of $\phi(h)(t)$ in $[0, 1/n]$. Then the above says that there are at least l eigenvalues of $\psi(h)(t)$ in $[0, 3/n]$. Continuing this way we have showed that the eigenvalues of $\phi(h)(t)$ and of $\psi(h)(t)$ can be paired to within $3/n$ one by one, in increasing order.

We now claim that that the eigenvalue maps of ϕ and ψ can be paired to within $3/n$ one by one in increasing order at each point of $[0, 1]$. For fixed $t \in (0, 1)$, since the numbers of the small eigenvalue maps are the same for ϕ and ψ, we may first pair them. For the rest, we group the same $\|\bar{n}_0\|$ eigenvalues together. Each such group corresponds to a point in $[0, 1]$ which is equal to one of $\alpha_i(t)$ or $\beta_j(t)$. If we order $\alpha_i(t) \leqslant \alpha_{i+1}(t)$ and $\beta_i(t) \leqslant \beta_{i+1}(t)$ for $i = 1, ..., p-1$, then

$$|\alpha_i(t) - \beta_i(t)| \leqslant 3/n \qquad i = 1, ..., p.$$

As a consequence, $\{\alpha_i(t)\}$ and $\{\beta_j(t)\}$ can be paired to within $3/n$ one by one for all $t \in (0, 1)$. By continuity, this is also true for $t = 0, 1$.

Deform each β_j to the corresponding α_i, we change ψ to a new map ψ_1. Now ψ_1 and ψ are within ε on F. It remains to show that ϕ and ψ_1 are arbitrarily close on F upto unitary equivalence. This is follows from the Proposition 4.1. ∎

5. PROOF OF THE MAIN THEOREM

The following basic results can be proved in a now standard way (cf. Section 4 of [EGJS] and the references therein):

5.1. LEMMA. *If an inductive limit A of finite direct sum of splitting interval algebras is unital and simple, but not finite dimensional, then there is an inductive sequence $\{A_m; \phi_{m,n}\}$, where each A_m is a finite direct sum of splitting interval algebras and each $\phi_{m,n}$ is an* injective *unital map of standard form, such that $A = \lim_{\to}(A_m, \phi_{m,n})$.*

To state the second result, we recall some basic definitions. Let A be a finite direct sum of splitting interval algebras and $\phi: A \to M_k(\mathbb{C})$ a C^*-homomorphism. Then ϕ is unitary equivalent, as a finite dimensional representation of A, to a finite direct sum of irreducible representations of A. Let $sp(\phi)$ be the subset of $sp(A)$ corresponding to these irreducible representations. Equivalently, $sp(\phi)$ is the finite (non-empty) subset of $sp(A)$ satisfying: for $f \in A$, $\phi(f) = 0$ if and only if $f|_{sp(\phi)} \equiv 0$.

5.2. LEMMA. *Let $(A_m, \phi_{m,n})$ be an inductive sequence where each algebra A_m is a finite direct sum of splitting interval algebras and each map $\phi_{m,n}$ is unital and injective. Then $A = \lim_{\to}(A_m, \phi_{m,n})$ is simple if, and only if for any nonempty open subset $\mathcal{O} \subseteq sp(A_m)$, there is an integer N such that for any $n > N$, and for any $x \in sp(A_n)$,*

$$sp(\phi^x_{m,n}) \cap \mathcal{O} \neq \emptyset,$$

where $\phi^x_{m,n}$ denotes the composition of $\phi_{m,n}$ with the canonical evaluation map $Q_x: B \to B_x$.

In particular, as $n \to +\infty$, for any $x \in sp(A_n)$, the size of the fibre A_x over x will eventually exceed any prescribed number. In fact, it follows immediately from Lemma 5.2 that:

5.3. COROLLARY. *Let $(A_m, \phi_{m,n})$ be as in Lemma 5.2. Then for any non-zero $e \in K_0^+(A_m)$ and any $K > 0$, there is an integer N such that for any $n > N$, each entry of $(\phi_{m,n})_*(e) \in K_0(A_n)$ will be greater than K.*

We now begin to prove the Main Theorem of this paper, as stated in the Introduction. Our proof will follow the strategy of Elliott of [E2]. We shall present below a proof for the homomorphism part, the isomorphism part being similar. Also, we shall be sketchy where the method in [E2] works without much change.

So let $A = lim_{\to}(A_m, \phi_{m,n})$ and $B = \lim_{\to}(B_m, \varphi_{m,n})$ be two simple inductive limits of finite direct sums of splitting interval algebras, together with a compatible pair $(\kappa; \theta)$ for $(A; B)$. If A or B is finite dimensional, the proof is straightforward (by Lemma 5.1 and comparison of K_0 as scaled ordered groups). From now on, we shall assume that both A and B are not finite dimensional and, by Lemma 5.1, each map $\phi_{m,n}$ and $\varphi_{m,n}$ is injective and of standard form.

5.4. *Approximate lifting of the compatible pair.* Since the positive cone in $K_0(A_m)$ for each m is finitely generated (cf. proof of Lemma 2.5), the homomorphism κ can be lifted (cf. Section 7 of [E1]): That is, after passing to subsequences and relabelling, there is a morphism $\kappa_m\colon K_0(A_m) \to K(B_m)$ for each m such that the following diagram commutes

$$\begin{tikzcd} K_0(A_1) \arrow[r, "(\phi_{1,2})_*"] \arrow[d, "\kappa_1"'] & K_0(A_2) \arrow[r, "(\phi_{2,3})_*"] \arrow[d, "\kappa_2"'] & K_0(A_3) \arrow[r] \arrow[d, "\kappa_3"'] & \cdots \\ K_0(B_1) \arrow[r, "(\varphi_{1,2})_*"] & K_0(B_2) \arrow[r, "(\varphi_{2,3})_*"] & K_0(B_3) \arrow[r] & \cdots. \end{tikzcd} \tag{5-1}$$

We note that each κ_m is faithful since each horizontal map in (5-1) is faithful.

The continuous affine map ψ_T can also be lifted, approximately, by approximating the spectra of A_m by finite points (see [E2] for details): That is, after passing to subsequences and relabelling, there are continuous affine maps $\theta_m\colon T(B_m) \to T(A_m)$ which make the following diagram approximately commute ([E2]),

$$\begin{tikzcd} T(A_1) & T(A_2) \arrow[l, "\phi_{1,2}^*"'] & T(A_3) \arrow[l, "\phi_{2,3}^*"'] & \cdots \arrow[l] \\ T(B_1) \arrow[u, "\theta_1"] & T(B_2) \arrow[l, "\varphi_{1,2}^*"'] \arrow[u, "\theta_2"] & T(B_3) \arrow[l, "\varphi_{2,3}^*"'] \arrow[u, "\theta_3"] & \cdots, \arrow[l] \end{tikzcd} \tag{5-2}$$

in the sense that there are finite sets $F_m \subseteq A_m$ such that $\phi_{m,n}(F_m) \subseteq F_n$, $\bigcup_m \phi_m(F_m)$ is dense in A, and

$$|\langle \phi_{m,m+1}(f), \theta_{m+1}(t) - \langle f, \theta_m \varphi_{m,m+1}^*(t)\rangle| < \frac{1}{2^m}, \tag{5-3}$$

for all $f \in F_m$ and $t \in T(B_{m+1})$. And, thanks to Proposition 2.7, each θ_m can be chosen to be compatible with κ_m.

5.5. *Implementation of κ_m and θ_m by a homomorphism.* We claim that, after passing to subsequences (and relabelling) one more time, κ_m and θ_m can be implemented by an injective unital ρ_m of standard form.

Indeed, for any prescribed finite subset $F_m \in A_m$, let $N = N(F_m; 2^{-(m+1)})$ be the constant in Theorem 3.7, and let $K = 2^{m+2} \cdot N \cdot \max\{\|f\| : f \in F_m\}$. It follows from Corollary 5.3 that all entries of $\kappa_m(e)$, where e is the unit of any direct summand of A_m, are greater than K. (If this is not true for κ_m, then by Corollary 5.3, we can find an n large enough such that this is true for $(\varphi_{m,n})_* \circ \kappa_m$. We then simply rename B_m). Then it follows from Lemmas 2.2, 2.3 and Theorem 3.7 that there is an injective homomorphism $\rho_m: A_m \to B_m$ of standard form such that ρ_m induces κ_m and

$$|\langle f, \theta_m(t)\rangle - \langle \rho(f), t\rangle| < \frac{1}{2^m},$$

for all $f \in F_m$ and $t \in T(B_m)$. In other words, there is a diagram,

$$\begin{array}{ccccccc} A_1 & \xrightarrow{\phi_{1,2}} & A_2 & \xrightarrow{\phi_{2,3}} & A_3 & \longrightarrow & \cdots \\ {\scriptstyle \rho_1}\downarrow & & {\scriptstyle \rho_2}\downarrow & & {\scriptstyle \rho_3}\downarrow & & \\ B_1 & \xrightarrow{\varphi_{1,2}} & B_2 & \xrightarrow{\varphi_{2,3}} & B_3 & \longrightarrow & \cdots, \end{array} \tag{5-4}$$

which induces diagram (5-1) (which is commutative) and diagram (5-2) (which is approximately commutative), respectively. However, diagram (5-4) itself might fail to be almost commutative, and therefore, fail to pass to the inductive limits. This will be fixed in the following paragraph.

5.5. *The existence of ρ.* Finally, the same arguments as in [E2], based on the injectivity of each homomorphism and the simplicity of A and B, can be used to show that, after passing to subsequences and relabelling, the condition (ii) in Theorem 4.2 is satisfied for any prescribed $n_m > 0$ and for any pair of maps $p_{m+1} \circ \phi_{m,m+1}$ and $\varphi_{m,m+1} \circ \rho_m$. Therefore, we can modify each ρ_m by an inner automorphism of B_m, in such a way that diagram (5-3) becomes approximately commutative, so that by Elliott's approximate intertwining argument (cf. Section 2 of [E1]) there is a *-homomorphism $\rho: A \to B$ inducing the corresponding maps between the K-groups and tracial state spaces.

6. AN EXAMPLE

As we mentioned in the introduction, the K_0-groups of the C^*-algebras we just classified may not have Riesz decomposition property. We shall now construct such an example A. A will be an inductive limit of splitting algebras. It is easy to check that a splitting algebra with at least two points at each end will not have Riesz decomposition property. We shall choose the size of the generic fibre of each splitting algebra as well as each connecting map carefully so that the limit A is simple and $K_0(A)$ does not have Riesz decomposition property.

Let n_k be an integer and let $A_k = \mathscr{S}(\bar{n}_0, \bar{n}_1)$ where $\bar{n}_x = (n_k, n_k)$ for $x = 0, 1$, i.e., the fibres of A_k at $x = 0, 1$ are $M_{n_k} \oplus M_{n_k}$. We shall call the first block the upper block and the second the lower block. The $K_0(A_k)$ can be identified as

$$K_0(A_k) = \{(x, y; a, b) \in \mathbb{Z}^4 \mid x + y = a + b\}$$

with the positive cone consisting of the elements with non-negative entries.

Next, we shall define a connecting map from A_k to A_{k+1}. The map will depend on two positive integers, say, m_k and l_k that we shall describe below. For any $f \in A_k$, $t \in [0, 1]$, define

$$\phi(m_k, l_k)(f)(t) =$$

$$A\, dU_k(t) \begin{pmatrix} f(\alpha_1(t)) & & & & & \\ & \ddots & & & & \\ & & f(\alpha_{m_k}(t)) & & & \\ & & & f(\beta_1(t)) & & \\ & & & & \ddots & \\ & & & & & f(\beta_{2l_k}(t)) \end{pmatrix}.$$

Here $\{\alpha_j(t)\}_{j=1}^{m_k}$ and $\{\beta_j(t)\}_{j=1}^{2l_k}$ are eigenvalue maps. They have the forms

$$\alpha_j(t) = t, \qquad j = 1, ..., m_k.$$

$$\beta_j(t) = \beta_{(l_k/2)+j} = (2j/l_k)\,t, \qquad j = 1, ..., l_k/2.$$

$$\beta_j(t) = \beta_{1l_k/2+j} = t + (1-t)\,2(j - l_k)/l_k, \qquad j = l_k + 1, ..., 3l_k/2.$$

We shall call each $\alpha_j(t)$ an identity map and each $\beta_j(t)$ a non-identity map. The above says that each non-identity map has an identical copy. Notice that l_k is even.

The map $\phi(m_k, l_k)$ so far has not been defined yet. We must specify $U_k(t)$ appeared in the definition. The unitary $U_k(t)$ is to ensure the following. Each upper block of $f(\alpha_j(0))$ or $f(\alpha_j(1))$ is sitting in the upper block of $A_{k+1}(x)$ for $x = 0, 1$, respectively. Similarly for the lower blocks. For $\beta_j(t)$, we ask that one $f(\beta_j(t))$, $t = 0, 1$ will be sitting in the upper blocks of $A_{k+1}(x)$ for $x = 0, 1$ and the other identical one is sitting in the lower blocks of $A_{k+1}(x)$ for $x = 0, 1$. In terms of formula we have

$$U_k(x) \begin{pmatrix} 0 & & \\ & f(\alpha_j(x)) & \\ & & 0 \end{pmatrix} U_k^*(x) = (*, *) \in A_{k+1}(x)$$

and

$$U_k(x)\begin{pmatrix}0 & & \\ & f(\beta_j(x)) & \\ & & 0\end{pmatrix}U_k^*(x)=(*,0) \qquad \text{or} \qquad (0,*)\in A_{k+1}(x)$$

for $x=0, 1$. Now $\phi(m_k, l_k)$ is well-defined. Hence, we have an C^*-algebra $A=\lim_{\to}(A_k, \phi(m_k, l_k))$. We remark that in order that $\phi(m_k, l_k)$ to be unital, n_{k+1} is determined once n_k is given.

Next, we analyze the composition $\phi(m_k, l_k)\circ\phi(m_{k+1}, l_{k+1})$. We claim that it has a similar form as those $\phi(m_j, l_j)$ with indexes

$$(m_k m_{k+1}, m_k l_{k+1}+l_k m_{k+1}+2l_k l_{k+1}).$$

except that the corresponding non-identity maps are not equally distributed. To see this, notice that an identity map compose with an identity map is again an identity map and all other combinations give non-identity maps. These new non-identity maps are sitting in the right blocks and they appear in pairs. The total numbers of the non-identity maps are $2\times(m_k l_{k+1}+l_k m_{k+1}+2l_k l_{k+1})$.

Let us now check that the inductive limit does have so called δ density (Lemma 5.2) which will ensure A to be simple. It is enough to look at the composition map from A_1 to A_k. Since each non-identity map of $\phi(m_k, l_k)$ composes with an identity map is the same non-identity map and since all m_j are positive, we conclude that the composition map at least have $2/l_k$ density. So if we choose $\{l_k\}$ to be a strictly increasing sequence, A must be simple. We shall fix this sequence $\{l_k\}$.

Finally, we shall choose $\{m_k\}$ to ensure that $K_0(A)$ does not have Riesz decomposition property. What we shall do is to pick up four elements in $K_0(A_1)$ such that they do not have Riesz decomposition property. We then choose m_k so that their images in all $K_0(A_k)$ also do not have Riesz decomposition property.

Let us fix four elements in $K_0(A_1)$ as follows:

$$e_1=(2,0,1,1), \qquad e_2=(1,1,1,1);$$

$$g_1=(1,0,1,0), \qquad g_2=(1,0,0,1).$$

It is evident that $e_i>g_j$ for $i, j=1, 2$. Since no element from the positive cone of $K_0(A_1)$ can be between $\min(e_1, e_2)$ and $\max(g_1, g_2)$, these four elements have no Riesz decomposition property. Let us first choose m_1. Under our convention, identity maps split at the ends of the intervals and non-identity maps do not split at the ends. Since rank $e_1 = \text{rank } e_2 = 4$ and rank $g_1 = \text{rank } g_2 = 2$, we have

$$K_0(\phi(m_1, l_1))(e_1) = m_1(2, 0, 1, 1) + 4l_1(1, 1, 1, 1),$$
$$K_0(\phi(m_1, l_1))(e_2) = m_1(1, 1, 1, 1) + 4l_1(1, 1, 1, 1),$$
$$K_0(\phi(m_1, l_1))(g_1) = m_1(1, 0, 1, 0) + 2l_1(1, 1, 1, 1),$$
$$K_0(\phi(m_1, l_1))(g_2) = m_1(1, 0, 0, 1) + 2l_1(1, 1, 1, 1),$$

Hence

$$\min(K_0(\phi(m_1, l_1))(e_1), K_0(\phi(m_1, l_1))(e_2))$$
$$= (m_1 + 4l_1, 4l_1, m_1 + 4l_1, m_1 + 4l_1)$$
$$\max(K_0(\phi(m_1, l_1))(g_1), K_0(\phi(m_1, l_1))(g_2))$$
$$= (m_1 + 2l_1, 2l_1, m_1 + 2l_1, m_1 + 2l_1).$$

Notice that these two elements are not in $K_0(A_2)$.

Let (x, y, s, t) be an positive element of $K_0(A_2)$ sitting between the above two elements. We should have

$$m_1 + 4l_1 \leqslant x + y \leqslant m_1 + 8l_1$$
$$2m_1 + 4l_1 \leqslant s + t \leqslant 2m_1 + 8l_1.$$

If we set $m_1 + 8l_1 < 2m_1 + 4l_1$, such (x, y, s, t) will not exist since $x + y = s + t$. The above inequality gives us $m_1 > 4l_1$. We shall fix such a m_1.

Next, we shall choose m_2. Recall that for any (m_2, l_2), the composition $\phi(m_1, l_1) \circ \phi(m_2, l_2)$ has the indexes $(m_1 m_2, l_1 m_2 + m_1 l_2 + 2l_1 l_2)$. According to our above computation, the images of the four elements in $K_0(A_3)$ will not have Riesz decomposition property if we choose

$$m_1 m_2 > 4(l_1 m_2 + m_1 l_2 + 2l_1 l_2)$$

or

$$m_1 > 4(l_1 + m_1 l_2/m_2 + 2l_1 l_2/m_2).$$

This can be achieved by choosing m_2 large. This process continues. Denote the two indexes of $\phi(m_1, l_1) \circ \phi + (m_2, l_2)$ by (m, l). We still have $m > 4l$. Hence m_3 can be chosen so that the images of the four elements in $K_0(A_4)$ do not have Riesz decomposition property. Continuing choosing $\{m_k\}$ this way, then the images of the four elements e_1, e_2, g_1 and g_2 in $K_0(A_k)$ fail to have Riesz decomposition property. This will prevent them to have Riesz decomposition property in $K_0(A)$. This completes our construction for the example.

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